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Observers for Descriptor Systems with Slope-restricted Nonlinearities

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Abstract: This paper addresses the observer design problem for a class of nonlinear descriptor systems whose nonlinear terms are slope-restricted. The full-order observer is formulated as a nonlinear descriptor system. A linear matrix inequality (LMI) condition is derived to construct the full-order observer. The existence and uniqueness of the solution to the obtained observer system are guaranteed. Furthermore, under the same LMI condition and a common assumption, a reduced-order observer is designed. Finally, the design methods are reduced to a strict LMI problem and illustrated by a numerical example.

Keywords: Descriptor systems, slope-restricted nonlinearities, observer, linear matrix inequality (LMI).

1 Introduction

Descriptor systems (also referred to as singular, semi-state, implicit, constrained, differential-algebraic equation, or generalized state-space systems) arise naturally as dynamic models of a wide range of engineering applications, such as Leontief dynamic model, chemical engineering systems, and electrical and mechanical[1−4]. Great efforts have been made to investigate observer design problems for descriptor systems. Many approaches have been developed for linear descriptor systems (see [5–12] and the references therein). In [6, 7], some full and reduced-order observer design methods for linear descriptor systems were derived from the generalized Sylvester equation. In [13, 14], a kind of generalized proportional-integral (PI) and proportional-integral-derivative (PID) observers was proposed for linear descriptor systems.

Observer design for nonlinear descriptor systems also attracts much attention, and some recent progresses are reported in [15–20]. Using coordinate transformation, a local asymptotic observer for nonlinear descriptor systems was designed in [15]. By dividing the system into dynamic system and static system, a full-order observer design method was presented in [16]. A class of nonlinear descriptor systems in quasi-linear form was considered in [17], where the given observer design method was based on rewriting the descriptor system as an equivalent system of (explicit) differential equations on a restricted manifold. In [18], an observer was designed for semi-explicit nonlinear descriptor systems of index one. The proposed observer was formulated as a differential-algebraic equation of index one, and the observer error dynamics was ensured to be locally stable. In [19], the authors considered descriptor systems with globally Lipschitz nonlinearities and external disturbances. The obtained observer design methods were reduced to linear matrix inequality (LMI) ones. In [20], the issues of full-order and reduced-order observer design were considered for a class of descriptor systems with global Lipschitz constraints. The designs of both types of observers were formulated into unified LMI conditions.

In practice, some Lipschitz nonlinearities satisfy certain slope restrictions. In these cases, the observer design methods for Lipschitz descriptor systems[19, 20] may be conservative. However, such a problem remained open. In this paper, we consider the observer design problem for descriptor systems with slope-restricted nonlinearities. Firstly, we construct a full-order observer which admits more free variables than those given by [20] and present an LMI-based design method that can be easily performed by using the existing tools. The obtained observer system is shown to have a unique solution for any compatible initial condition. Then, we design a reduced-order observer under the same LMI condition and a common assumption. Finally, we reduce the design method to a strict LMI problem and present an illustrative example.

2 Preliminaries

In this section, notations employed in this paper and terminology on descriptor systems are given. We use $\mathbf{R}$ to denote the set of real numbers. $\mathbf{R}^n$ and $\mathbf{R}^{n_1\times n_2}$ are the obvious extensions to vectors and matrices of the specified dimensions. Let I or I_r denote the identity matrix with appropriate dimension. The notation $A > 0$ (respectively $A < 0$) means that the matrix A is symmetric and positive definite (respectively negative definite). $A \geqslant 0$ (respectively $A \leqslant 0$) means that the matrix A is symmetric and semi-positive definite (respectively semi-negative definite). A^{T} is the matrix transpose of A. $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote the real part and the image part of a complex number, respectively. $\det(\cdot)$ denotes the determinant of a matrix. $\deg(\cdot)$ represents the degree of a polynomial. $\|\cdot\|$ denotes the Euclidean norm of a vector or matrix. “$\star$” denotes an element induced by transpose. “ := ” stands for an equality

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by definition.

Consider a linear time-invariant descriptor system

$$E\dot{x} = Ax \tag{1}$$

where $x \in \mathbf{R}^n$ is the state variable, the matrices $A, E \in \mathbf{R}^{n\times n}$ are constant and $\mathrm{rank}(E) = s \leqslant n$.

First, we state here some basic definitions which will be used in the sequel and can be found in [1, 3]. If $\det(sE - A) \neq 0$ for some complex number s, then the pair (E, A) is said to be regular. A regular pair (E, A) is called impulse-free if $\deg(\det(sE - A)) = \mathrm{rank}(E)$. If all roots of $\det(sE - A) = 0$ lie in $\mathrm{Re}(s) < 0$, then (E, A) is called stable. The pair (E, A) is said to be admissible if it is regular, impulse-free, and stable. It is proved in [2] that (E, A) is regular and impulse-free if and only if there exist two nonsingular matrices M and N such that (E, A) can be transformed into the Weierstrass canonical form

$$MEN = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}, \quad MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-s} \end{bmatrix} \tag{2}$$

where $A_1 \in \mathbf{R}^{s\times s}$.

Lemma 1[21]**.** The pair (E, A) is admissible if and only if there exists a matrix $X \in \mathbf{R}^{n\times n}$ satisfying

$$E^{\mathrm{T}}X = X^{\mathrm{T}}E \geqslant 0$$
$$A^{\mathrm{T}}X + X^{\mathrm{T}}A < 0.$$

3 Main results

Consider the descriptor system

$$\begin{aligned} E\dot{x} &= Ax + B\gamma(t, Hx) + \psi(t, y, u) \\ y &= Cx \end{aligned} \tag{3}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $y \in \mathbf{R}^p$ are the system state, input, and output, respectively. The system matrices E, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times r}$, $C \in \mathbf{R}^{p\times n}$, $H \in \mathbf{R}^{r\times n}$ are constant and $\mathrm{rank}(E) = s \leqslant n$. ψ is a locally Lipschitz function. The nonlinear vector function $\gamma(t, Hx)$ is continuous with respect to its variables and defined as

$$\gamma(t, Hx) = \begin{bmatrix} \gamma_1(t, Hx) & \gamma_2(t, Hx) & \cdots & \gamma_r(t, Hx) \end{bmatrix}^{\mathrm{T}} \tag{4}$$

where $\gamma_i(t, Hx) = \gamma_i(t, \Sigma_{j=1}^n(H_{ij}x_j)), i = 1, 2, \cdots, r$.

The nonlinear term γ is assumed to satisfy

$$\begin{aligned} &0 \leqslant \frac{\gamma_i(t, a) - \gamma_i(t, b)}{a - b} \leqslant \lambda_i, \\ &\forall a, b \in \mathbf{R},\ a \neq b,\ i = 1, 2, \cdots, r,\ \forall t \geqslant 0 \end{aligned} \tag{5}$$

where $\lambda_i > 0$ is known. Let $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_r\}$.

For system (3), we construct a full-order observer of the form

$$\begin{aligned} E\dot{\hat{x}} = \ & A\hat{x} + L(C\hat{x} - y) + B\gamma(t, H\hat{x} + K(C\hat{x} - y)) + \\ & \psi(t, y, u) \end{aligned} \tag{6}$$

where matrices $L \in \mathbf{R}^{n\times p}$ and $K \in \mathbf{R}^{r\times p}$ are to be designed.

Remark 1. For standard state-space systems, observer (6) with $E = I_n$ has been considered in [22–24]. The full-order observer of [20] is a special case of (6) with $K = 0$.

Defining the state estimation error by $e = x - \hat{x}$, from (3) and (6), the dynamics of the observer error $e = x - \hat{x}$ is governed by

$$E\dot{e} = (A + LC)e + B\phi \tag{7}$$

where $\phi = \gamma(t, v) - \gamma(t, w)$, $v = Hx$, $w = H\hat{x} + K(C\hat{x} - y)$.

From the continuity of $\gamma(t, Hx)$ and the slope restriction (5), it follows that

$$\gamma_i(t, v_i) - \gamma_i(t, w_i) = \delta_i(t)(v_i - w_i)$$

where $0 \leqslant \delta_i(t) \leqslant \lambda_i$. Then, for any real number $\tau > 0$, it holds that

$$\begin{aligned} &\tau[\gamma_i(t, v_i) - \gamma_i(t, w_i)]^2 = \\ &\quad \tau\delta_i(t)[\gamma_i(t, v_i) - \gamma_i(t, w_i)](v_i - w_i) \leqslant \\ &\quad \tau\lambda_i[\gamma_i(t, v_i) - \gamma_i(t, w_i)](v_i - w_i). \end{aligned} \tag{8}$$

Letting $\Delta(t) = \mathrm{diag}\{\delta_1(t), \delta_2(t), \cdots, \delta_r(t)\}$, from (8), we have

$$\phi = \Delta(t)(H + KC)e.$$

Then

$$\phi^{\mathrm{T}}\Gamma\phi \leqslant \phi^{\mathrm{T}}\Gamma\Lambda(H + KC)e \tag{9}$$

for any diagonal matrix $\Gamma \in \mathbf{R}^r$ with $\Gamma > 0$.

The following theorem describes a full-order observer design method.

Theorem 1. If there exist matrices $P \in \mathbf{R}^{n\times n}$, $Q \in \mathbf{R}^{n\times p}$, $F \in \mathbf{R}^{p\times r}$, and diagonal matrix $\Gamma \in \mathbf{R}^r$ with $\Gamma > 0$ satisfying

$$E^{\mathrm{T}}P = P^{\mathrm{T}}E \geqslant 0 \tag{10}$$

$$\begin{bmatrix} A^{\mathrm{T}}P + P^{\mathrm{T}}A + C^{\mathrm{T}}Q^{\mathrm{T}} + QC & \star \\ B^{\mathrm{T}}P + \Gamma\Lambda H + \Lambda FC & -2\Gamma \end{bmatrix} < 0 \tag{11}$$

then there exists an observer of the form (6) for system (3), and the resulting observer gains $L = P^{-\mathrm{T}}Q$, $K = \Gamma^{-1}F$ ensure that the estimation error is exponentially convergent.

Proof. Assume that LMIs (10) and (11) are feasible and P is nonsingular.

Let $L = P^{-\mathrm{T}}Q$, $K = \Gamma^{-1}F$, $\overline{A} = A + LC$, $\overline{C} = H + KC$. Then, LMI (11) becomes

$$\begin{bmatrix} \overline{A}^{\mathrm{T}}P + P^{\mathrm{T}}\overline{A} & \star \\ B^{\mathrm{T}}P + \Gamma\Lambda\overline{C} & -2\Gamma \end{bmatrix} < 0. \tag{12}$$

Define the following quadratic Lyapunov function:

$$V(e) = e^{\mathrm{T}}E^{\mathrm{T}}Pe. \tag{13}$$

Calculating the derivative of V along the trajectories of system (7) and using inequality (9), we have

$$\begin{aligned} \dot{V}(e)|_{(7)} = \ & \dot{e}^{\mathrm{T}}E^{\mathrm{T}}Pe + e^{\mathrm{T}}P^{\mathrm{T}}E\dot{e} = \\ & (\overline{A}e + B\phi)^{\mathrm{T}}Pe + e^{\mathrm{T}}P^{\mathrm{T}}(\overline{A}e + B\phi) \leqslant \\ & e^{\mathrm{T}}(\overline{A}^{\mathrm{T}}P + P^{\mathrm{T}}\overline{A})e + 2e^{\mathrm{T}}P^{\mathrm{T}}B\phi - \\ & 2\phi^{\mathrm{T}}\Gamma\phi + 2\phi^{\mathrm{T}}\Gamma\Lambda\overline{C}e = \\ & \begin{bmatrix} e \\ \phi \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \overline{A}^{\mathrm{T}}P + P^{\mathrm{T}}\overline{A} & \star \\ B^{\mathrm{T}}P + \Gamma\Lambda\overline{C} & -2\Gamma \end{bmatrix} \begin{bmatrix} e \\ \phi \end{bmatrix}. \end{aligned} \tag{14}$$

By LMI (12), there exists $\alpha_1 > 0$ such that

$$\dot{V}(e)|_{(7)} \leqslant -\alpha_1 e^{\mathrm{T}} e.$$

Denote $\alpha_2 = \lambda_{\max}(E^{\mathrm{T}}P)$. Then, $\alpha_2 > 0$.

As a result,

$$\dot{V} \leqslant -\alpha_1\alpha_2^{-1}V$$

which implies that

$$V \leqslant \mathrm{e}^{-\mu t}V(e(0)) \tag{15}$$

where $\mu = \alpha_1\alpha_2^{-1}$.

By Lemma 1, LMIs (10) and (12) indicate that $(E, \overline{A})$ is admissible. Then, there exist two nonsingular matrices $M, N \in \mathbf{R}^{n\times n}$ such that

$$MEN = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}, \quad M\overline{A}N = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-s} \end{bmatrix}$$

where $A_1 \in \mathbf{R}^{s\times s}$. Correspondingly, we partition MB, ML, and $\overline{C}N$ into

$$MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ ML = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \ \overline{C}N = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

and let

$$M^{-\mathrm{T}}PN = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

From inequalities (10) and (12), it follows that $P_{11} = P_{11}^{\mathrm{T}} > 0$ and $P_{12} = 0$.

Pre- and post-multiplying (12) by

$$\begin{bmatrix} N^{\mathrm{T}} & 0 \\ 0 & I_r \end{bmatrix}$$

and its transpose, respectively, yields

$$\begin{bmatrix} A_1^{\mathrm{T}}P_{11} + P_{11}A_1 & \star & \star \\ P_{21} & P_{22} + P_{22}^{\mathrm{T}} & \star \\ B_1^{\mathrm{T}}P_{11} + B_2^{\mathrm{T}}P_{21} + \Gamma\Lambda C_1 & B_2^{\mathrm{T}}P_{22} + \Gamma\Lambda C_2 & -2\Gamma \end{bmatrix} < 0. \tag{16}$$

Pre- and post-multiplying (16) by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & -B_2^{\mathrm{T}} & I \\ 0 & I & 0 \end{bmatrix}$$

and its transpose, respectively, gives

$$\begin{bmatrix} A_1^{\mathrm{T}}P_{11} + P_{11}A_1 & \star & \star \\ B_1^{\mathrm{T}}P_{11} + \Gamma\Lambda C_1 & -\Gamma\Lambda C_2B_2 - B_2^{\mathrm{T}}C_2^{\mathrm{T}}\Lambda\Gamma - 2\Gamma & \star \\ P_{21} & -P_{22}B_2 + C_2^{\mathrm{T}}\Lambda\Gamma & P_{22} + P_{22}^{\mathrm{T}} \end{bmatrix} < 0 \tag{17}$$

which implies

$$-\Gamma\Lambda C_2B_2 - B_2^{\mathrm{T}}C_2^{\mathrm{T}}\Lambda\Gamma - 2\Gamma < 0. \tag{18}$$

Letting $N^{-1}e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$, we have $V(e) = e^{\mathrm{T}}E^{\mathrm{T}}Pe = e_1^{\mathrm{T}}P_{11}e_1$. In addition, denoting $\alpha_3 = \lambda_{\min}(P_{11})$, it is easy to show that $\alpha_3 > 0$. Hence, it follows from (15) that

$$\|e_1\| \leqslant \sqrt{\alpha_3^{-1}V(0)}\,\mathrm{e}^{-\frac{1}{2}\mu t}. \tag{19}$$

Pre-multiplying (7) by M yields the restricted equivalent form of system (7)

$$\begin{aligned} \dot{e}_1 &= A_1e_1 + B_1\phi \\ 0 &= e_2 + B_2\phi. \end{aligned} \tag{20}$$

Then, using inequality (9), we have

$$\begin{aligned} \phi^{\mathrm{T}}\Gamma\phi \leqslant \phi^{\mathrm{T}}\Gamma\Lambda(C_1e_1 + C_2e_2) \leqslant \\ \phi^{\mathrm{T}}\Gamma\Lambda C_1e_1 - \phi^{\mathrm{T}}\Gamma\Lambda C_2B_2\phi. \end{aligned} \tag{21}$$

From (18) and (21), there exists some $\rho > 0$ such that

$$\|\phi\| \leqslant \rho\|e_1\| \tag{22}$$

which indicates

$$\|e_2\| \leqslant \rho\|B_2\|\|e_1\| \leqslant \rho\|B_2\sqrt{\alpha_3^{-1}V(0)}\,\mathrm{e}^{-\frac{1}{2}\mu t} \tag{23}$$

by inequality (19).

Therefore, the error state $e = N\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ is exponentially stable. □

Remark 2. Without loss of generality, we can assume that the matrix P satisfying LMIs (10) and (11) is nonsingular. Otherwise, there exists a small perturbation ΔP such that $P+\Delta P$ is nonsingular and satisfies (10) and (11). Such a discussion is common for design problems of descriptor systems (see, for example, [20]).

Remark 3. It can be seen that the slope-restricted nonlinearity $\gamma(t, Hx)$ satisfies Lipschitz condition. For Lipschitz descriptor systems, Lu and Ho[20] presented an LMI-based approach to construct observer of the form (6) with $K = 0$. Thus, our design method admits more free variables than those given by [20]. This is useful for design problems.

The following theorem will show that observer system (6) constructed by Theorem 1 has a unique solution for any given compatible initial condition.

Theorem 2. For any given compatible initial condition, there exists a unique solution to observer system (6), if there exist matrices $P \in \mathbf{R}^{n\times n}, Q \in \mathbf{R}^{n\times p}$, $F \in \mathbf{R}^{p\times r}$ and diagonal matrix $\Gamma \in \mathbf{R}^r$ with $\Gamma > 0$ satisfying LMIs (10) and (11).

Proof. We assume that LMIs (10) and (11) are feasible and use the notations in the proof of Theorem 1.

Let

$$N^{-1}\hat{x} = \begin{bmatrix} x_s \\ x_f \end{bmatrix}, \quad M\psi(t,y,u) = \begin{bmatrix} \psi_1(t,y,u) \\ \psi_1(t,y,u) \end{bmatrix}. \tag{24}$$

Then, pre-multiplying (6) by M yields the restricted equivalent form of system (6)

$$\begin{aligned} \dot{x}_s &= A_1x_s + B_1\gamma(t, C_1x_s + C_2x_f - Ky) - L_1y + \\ &\quad \psi_1(t,y,u) \\ 0 &= x_f + B_2\gamma(t, C_1x_s + C_2x_f - Ky) - L_2y + \\ &\quad \psi_2(t,y,u). \end{aligned} \tag{25}$$

For any given $x_{1s}, x_{2s} \in \mathbf{R}^s$, we assume that (x_{1s}, x_{1f}) and (x_{2s}, x_{2f}) satisfy the second equation of (25). Then, using inequality (9), we have

$$\Pi^{\mathrm{T}}\Gamma\Pi \leqslant \Pi^{\mathrm{T}}\Gamma\Lambda[C_1(x_{1s}-x_{2s})+C_2(x_{1f}-x_{2f})] = \Pi^{\mathrm{T}}\Gamma\Lambda C_1(x_{1s}-x_{2s}) - \Pi^{\mathrm{T}}\Gamma\Lambda C_2 B_2\Pi \tag{26}$$

where $\Pi = \gamma(t, C_1x_{1s}+C_2x_{1f}-Ky) - \gamma(t, C_1x_{2s}+C_2x_{2f}-Ky)$.

Inequality (26) shows

$$\Pi^{\mathrm{T}}(\Gamma + \Gamma\Lambda C_2 B_2)\Pi \leqslant \Pi^{\mathrm{T}}\Gamma\Lambda C_1(x_{1s}-x_{2s}). \tag{27}$$

By (18) and (27), there exists $\alpha > 0$ such that

$$\|\Pi\| \leqslant \alpha\|x_{1s}-x_{2s}\|. \tag{28}$$

Then, γ is Lipschitz with respect to x_s.

For the case where $x_{1s} = x_{2s}$, inequality (28) guarantees that $\Pi = 0$. Then, by the second equation of (25), we have

$$x_{1f} - x_{2f} = -B_2\Pi = 0$$

which shows that there exists a unique solution x_f for the second equation of (25) in terms of x_s, u, and y.

In addition, since γ is Lipschitz with respect to x_s, the solution to the dynamic equation of (25) exists and is unique. Hence, the solution $\hat{x} = N\begin{bmatrix} x_s \\ x_f \end{bmatrix}$ to system (6) exists and is unique. □

In practice, it may be more convenient to employ a reduced-order observer. To this end, we make the following common assumption.

Assumption 1. $\operatorname{rank}\begin{bmatrix} E^{\mathrm{T}} & C^{\mathrm{T}} \end{bmatrix} = n$ and $C = \begin{bmatrix} I_p & 0 \end{bmatrix}$.

Let $M_1 := E\begin{bmatrix} 0 \\ I_{n-p} \end{bmatrix}$. It is easy to show that $\operatorname{rank}(M_1) = n-p$ which guarantees that one can find a matrix $M_0 \in \mathbf{R}^{n\times p}$ such that

$$M := \begin{bmatrix} M_0 & M_1 \end{bmatrix}^{-1} \tag{29}$$

exists. Then, ME and MA are of the following structure:

$$ME = \begin{bmatrix} E_1 & 0 \\ E_2 & I_{n-p} \end{bmatrix}, \quad MA = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{30}$$

where $E_1 \in \mathbf{R}^{p\times p}$, $E_2 \in \mathbf{R}^{(n-p)\times p}$, $A_{11} \in \mathbf{R}^{p\times p}$, $A_{12} \in \mathbf{R}^{p\times(n-p)}$, $A_{21} \in \mathbf{R}^{(n-p)\times p}$, and $A_{22} \in \mathbf{R}^{(n-p)\times(n-p)}$.

We introduce the new state

$$\omega = \begin{bmatrix} 0 & I_{n-p} \end{bmatrix} x + (L_rE_1 + E_2)y$$

where $L_r \in \mathbf{R}^{(n-p)\times p}$ is the observer gain to be designed. For convenience, let

$$\Gamma_1 = \begin{bmatrix} I_p & 0 \\ -E_2 - L_rE_1 & I_{n-p} \end{bmatrix}.$$

Then, we have

$$x = \Gamma_1\begin{bmatrix} y \\ \omega \end{bmatrix} \tag{31}$$

and

$$\omega = \begin{bmatrix} L_r & I_{n-p} \end{bmatrix} MEx. \tag{32}$$

It can be seen from (31) that x can be easily estimated if only we can obtain the estimator of ω. The dynamics of state ω can be represented by

$$\begin{aligned} \dot{\omega} \;=\; & \begin{bmatrix} L_r & I_{n-p} \end{bmatrix} ME\dot{x} = \\ & \begin{bmatrix} L_r & I_{n-p} \end{bmatrix} M(Ax + B\gamma(t, Hx) + \psi(t, y, u)). \end{aligned} \tag{33}$$

Then, we have

$$\begin{aligned} &\dot{\omega} = (A_{22} + L_rA_{12})\omega + \begin{bmatrix} L_r & I_{n-p} \end{bmatrix} MB\times \\ &\gamma(t, (H_1 - H_2(E_2 + L_rE_1))y + H_2\omega) + \widetilde{\psi}(t, y, u) \end{aligned} \tag{34}$$

where $\widetilde{\psi}(t, y, u) = \begin{bmatrix} L_r & I_{n-p} \end{bmatrix} MB\psi(t, y, u) + [A_{21} + L_rA_{11} - (A_{22} + L_rA_{12})(E_2 + L_rE_1)]y$ and $H_1 \in \mathbf{R}^{r\times p}$, $H_2 \in \mathbf{R}^{r\times(n-p)}$ such that $H = \begin{bmatrix} H_1 & H_2 \end{bmatrix}$.

For system (34), the observer is given by

$$\begin{aligned} &\dot{\hat{\omega}} = (A_{22} + L_rA_{12})\hat{\omega} + \begin{bmatrix} L_r & I_{n-p} \end{bmatrix} MB\times \\ &\gamma(t, (H_1 - H_2(E_2 + L_rE_1))y + H_2\hat{\omega}) + \widetilde{\psi}(t, y, u). \end{aligned} \tag{35}$$

Letting $\delta = \omega - \hat{\omega}$, we have

$$\dot{\delta} = (A_{22} + L_rA_{12})\delta + \begin{bmatrix} L_r & I_{n-p} \end{bmatrix} MB\phi(t, z) \tag{36}$$

where $\phi(t, z) = \gamma(t, (H_1 - H_2(E_2 + L_rE_1))y + H_2\omega) - \gamma(t, (H_1 - H_2(E_2 + L_rE_1))y + H_2\hat{\omega})$ and $z = H_2\delta$.

Similarly to the derivation of inequality (9), it follows from condition (5) that

$$\phi^{\mathrm{T}}\Gamma\phi \leqslant \phi^{\mathrm{T}}\Gamma\Lambda H_2\delta \tag{37}$$

holds for arbitrary diagonal matrix $\Gamma > 0$.

Based on the above discussion, we describe a reduced-order observer design method in the following theorem.

Theorem 3. Under Assumption 1, if the conditions of Theorem 1 hold, then there exists a reduced-order observer in the form of (35) for system (3), and the estimation error is exponentially convergent.

Proof. Assume that the conditions of Theorem 1 hold; that is, inequalities (10) and (11) are feasible, and P is nonsingular.

Let

$$L = P^{-\mathrm{T}}Q,\ K = \Gamma^{-1}F,\ M^{-\mathrm{T}}P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$$

where M is defined as (29). Based on the new decomposition of matrices E and A in (30), inequality (10) shows

$$\begin{bmatrix} E_1^{\mathrm{T}}P_1 + E_2^{\mathrm{T}}P_3 & E_1^{\mathrm{T}}P_2 + E_2^{\mathrm{T}}P_4 \\ P_3 & P_4 \end{bmatrix} = \begin{bmatrix} P_1^{\mathrm{T}}E_1 + P_3^{\mathrm{T}}E_2 & P_3^{\mathrm{T}} \\ P_2^{\mathrm{T}}E_1 + P_4^{\mathrm{T}}E_2 & P_4^{\mathrm{T}} \end{bmatrix} \geqslant 0 \tag{38}$$

which implies that

$$\begin{bmatrix} E_1^{\mathrm{T}}P_1 + E_2^{\mathrm{T}}P_3 & P_3^{\mathrm{T}} \\ P_3 & P_4 \end{bmatrix} \geqslant 0. \tag{39}$$

Since P is nonsingular, $\begin{bmatrix} P_3 & P_4 \end{bmatrix}$ is a full row rank. Then, it follows from (39) that $P_4 > 0$.

Substituting (30) into (11) yields

$$\begin{bmatrix} \Gamma_2 + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} & \star \\ B^{\mathrm{T}}M^{\mathrm{T}} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} + \Gamma\Lambda \begin{bmatrix} H_1 + K & H_2 \end{bmatrix} & -2\Gamma \end{bmatrix} < 0 \tag{40}$$

where $\Gamma_2 = \begin{bmatrix} I_p \\ 0 \end{bmatrix} L^{\mathrm{T}}P + P^{\mathrm{T}}L \begin{bmatrix} I_p & 0 \end{bmatrix}$, which implies that

$$\begin{bmatrix} A_{12}^{\mathrm{T}}P_2 + A_{22}^{\mathrm{T}}P_4 + P_2^{\mathrm{T}}A_{12} + P_4A_{22} & \star \\ B^{\mathrm{T}}M^{\mathrm{T}} \begin{bmatrix} P_2 \\ P_4 \end{bmatrix} + \Gamma\Lambda H_2 & -2\Gamma \end{bmatrix} < 0. \tag{41}$$

Choosing $L_r = P_4^{-1}P_2^{\mathrm{T}}$, inequality (41) is equivalent to

$$S := \begin{bmatrix} P_4(A_{22} + L_rA_{12}) + (A_{22} + L_rA_{12})^{\mathrm{T}}P_4 & \star \\ B^{\mathrm{T}}M^{\mathrm{T}} \begin{bmatrix} L_r^{\mathrm{T}} \\ I_{n-p} \end{bmatrix} P_4 + \Gamma\Lambda H_2 & -2\Gamma \end{bmatrix} < 0. \tag{42}$$

Let the Lyapunov function be

$$V(\delta) = \delta^{\mathrm{T}}P_4\delta. \tag{43}$$

Calculating the derivative of $V(\delta)$ along the solutions of system (36) and using inequality (37), we have

$$\begin{aligned} \dot{V}|_{(36)} = {} & 2\delta^{\mathrm{T}}P_4(A_{22} + L_rA_{12})\delta + \\ & 2\delta^{\mathrm{T}}P_4 \begin{bmatrix} L_r & I_{n-p} \end{bmatrix} MB\phi \leqslant \\ & 2\delta^{\mathrm{T}}P_4(A_{22} + L_rA_{12})\delta + \\ & 2\delta^{\mathrm{T}}P_4 \begin{bmatrix} L_r & I_{n-p} \end{bmatrix} MB\phi - \\ & 2\phi^{\mathrm{T}}\Gamma\phi + 2\phi^{\mathrm{T}}\Gamma\Lambda H_2\delta = \\ & \begin{bmatrix} \delta^{\mathrm{T}} & \phi^{\mathrm{T}} \end{bmatrix} S \begin{bmatrix} \delta \\ \phi \end{bmatrix}. \end{aligned} \tag{44}$$

From (42) and (44), we have

$$V|_{(36)} < 0, \quad \forall \delta \neq 0.$$

Then, error system (36) is exponentially stable.

Then, under Assumption 1, if the conditions of Theorem 1 hold, then there exists a reduced-order observer in the form of (35) for system (3), and the estimation error is exponentially convergent. □

Using Theorems 1 and 3, both types of full-order and reduced-order nonlinear observers are constructed by a unified LMI approach. It is noted that LMIs (10) and (11) cannot be directly solved by the LMI toolbox of Matlab[25]. This motivates us to make some further improvements to convert them into strict LMIs. Let $E_{\perp} \in \mathbf{R}^{(n-s)\times n}$ satisfy $E_{\perp}E = 0$ and $\mathrm{rank}(E_{\perp}) = n - s$. According to [26], LMIs (10) and (11) are feasible if and only if there exist a positive-definite matrix $X \in \mathbf{R}^{n\times n}$, two matrices $Y \in \mathbf{R}^{(n-s)\times n}$ and $Q \in \mathbf{R}^{n\times p}$, and diagonal matrix $\Gamma \in \mathbf{R}^r$ with $\Gamma > 0$ satisfying

$$\begin{bmatrix} A^{\mathrm{T}}(XE + E_{\perp}^{\mathrm{T}}Y) + (XE + E_{\perp}^{\mathrm{T}}Y)^{\mathrm{T}}A + C^{\mathrm{T}}Q^{\mathrm{T}} + QC & \star \\ B^{\mathrm{T}}(XE + E_{\perp}^{\mathrm{T}}Y) + \Gamma\Lambda H + \Lambda FC & -2\Gamma \end{bmatrix} < 0. \tag{45}$$

Then, from Theorems 1 and 3, we have the following theorem that describes a unified strict LMI-based observer design method.

Theorem 4. If there exist a positive definite matrix $X \in \mathbf{R}^{n\times n}$, $F \in \mathbf{R}^{p\times r}$, two matrices $Y \in \mathbf{R}^{(n-s)\times n}$ and $Q \in \mathbf{R}^{n\times p}$, and diagonal matrix $\Gamma \in \mathbf{R}^r$ with $\Gamma > 0$ satisfying (45), then there exists a full-order observer of the form (6) with observer gains $L = P^{-\mathrm{T}}Q$, $P = XE + E_{\perp}^{\mathrm{T}}Y$, $K = \Gamma^{-1}F$.

If, further, Assumption 1 holds, then there exists a reduced-order observer of the form (35) with $L_r = (T_1)^{-1}(T_2)^{\mathrm{T}}$, where

$$T_1 = \begin{bmatrix} 0 & I_{n-p} \end{bmatrix} M^{-\mathrm{T}}P \begin{bmatrix} 0 & I_{n-p} \end{bmatrix}^{\mathrm{T}}$$

$$T_2 = \begin{bmatrix} I_p & 0 \end{bmatrix} M^{-\mathrm{T}}P \begin{bmatrix} 0 & I_{n-p} \end{bmatrix}^{\mathrm{T}}.$$

4 An example

Consider system (3) with

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and

$$\gamma(t, Hx) = \begin{bmatrix} 0.5(\sin(x_1) + x_1) \\ 0.5(\sin(2x_3) + 2x_3) \end{bmatrix}.$$

Then, it can be verified that Assumption 1 holds, and the nonlinearity γ satisfies the slope restriction (5) with

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We will design full and reduced-order observers for this example by using Theorem 4.

Letting $E_{\perp} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and solving the LMIs in Theorem 4, we have

$$\boldsymbol{X} = \begin{bmatrix} 15.5652 & -12.3878 & -2.1210 \\ -12.3878 & 11.2315 & 1.9108 \\ -2.1210 & 1.9108 & 0.3297 \end{bmatrix},$$

$$\boldsymbol{Y} = \begin{bmatrix} -32.1170 & 13.1776 & -10.9586 \end{bmatrix},$$

$$\boldsymbol{Q} = \begin{bmatrix} 18.7115 \\ -11.0567 \\ 41.1648 \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} 62.3231 \\ -32.1170 \end{bmatrix},$$

$$\boldsymbol{\Gamma} = \begin{bmatrix} 1.9108 & 0 \\ 0 & 0.3297 \end{bmatrix}.$$

Then

$$\boldsymbol{P} = \begin{bmatrix} 0 & 15.5652 & -12.3878 \\ 0 & -12.3878 & 11.2315 \\ -32.1170 & 11.0567 & -9.0478 \end{bmatrix}.$$

Thus, the full-order observer gains are given by

$$L = \begin{bmatrix} 18.3881 \\ 23.4771 \\ -0.5826 \end{bmatrix}, \quad K = \begin{bmatrix} 32.6162 \\ -97.4174 \end{bmatrix}.$$

The simulation results given in Fig. 1 show that the trajectories of the original system can be estimated by those of the obtained observer system, and the error is exponentially convergent.

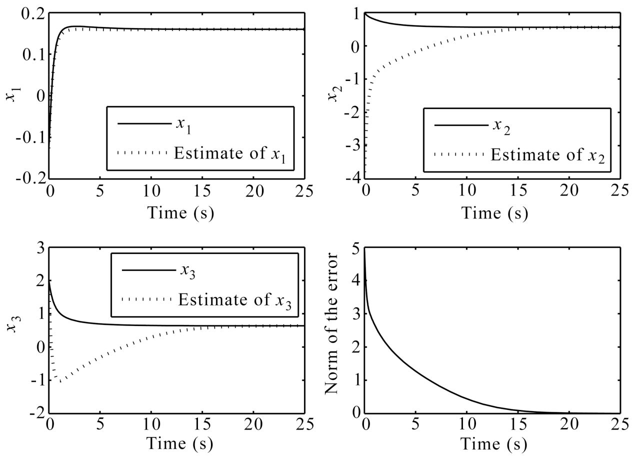

Fig. 1 Estimation performance of the full-order observer

To get the reduced-order observer, by simple computation, we have

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus, we can choose

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then

$$E_1 = 0, \quad E_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_{11} = 1,$$

$$A_{12} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -2 & 0 \\ 2 & -2 \end{bmatrix}$$

and

$$M^{-\mathrm{T}} P = P.$$

Then, the reduced-order observer gain is given by

$$L_r = \begin{bmatrix} 0.1403 \\ 1.5405 \end{bmatrix}.$$

Fig. 2 shows the trajectories of x_2 and x_3 and their estimates. It can be seen that the error is exponentially convergent.

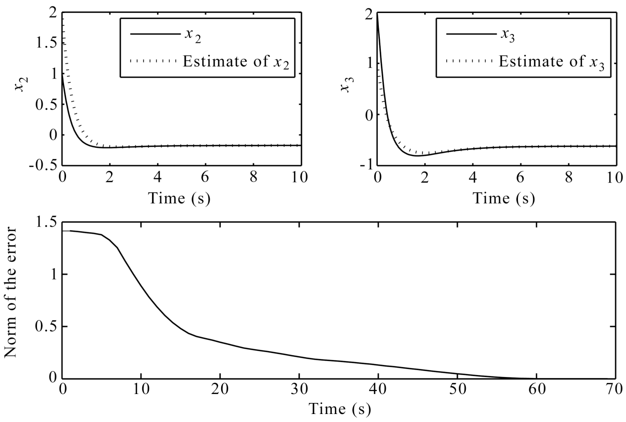

Fig. 2 Estimation performance of the reduced-order observer

On the other hand, it can be seen that the nonlinearity γ satisfies the global Lipschitz condition. However, with the aid of Matlab 6.5, we find that the observer design methods for Lipschitz descriptor systems given by [20] are not feasible for this example. This shows the advantage of our newly developed methods.

5 Conclusions

In this paper, we have studied the observer design problem of a class of nonlinear descriptor systems. The nonlinear terms are slope-restricted. Both types of full-order and reduced-order nonlinear observers are constructed by a unified LMI approach, by which the observer error system is guaranteed to be exponentially convergent. The example has illustrated the validity of the obtained methods.

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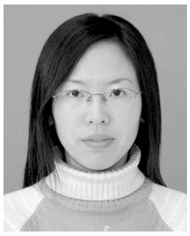

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